Synchronization and desynchronization of weakly coupled oscillators

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We study the dynamics of coupled nonlinear oscillators and investigate the conditions under which linear coupling of these oscillators leads to either synchronization or desynchronization of the relative phases of oscillations. This question has been previously addressed for infinitesimally small limit cycle oscillators, i.e., for oscillators in the vicinity of a Hopf bifurcation. In the present paper, we generalize these results by studying limit cycle oscillators with finite size. This is achieved by expanding the dynamics of the nonlinear dynamical systems in terms of deviations from a circular limit cycle of finite size. Taking into account only lowest-order expansion terms, we analytically derive a condition for desynchronization which goes beyond the Benjamin-Feir criterion obtained in the framework of the third-order Ginzburg-Landau theory. [S1063-651X(97)02408-2]

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The behavior of interacting limit cycle oscillators is fundamental to a number of problems such as spatiotemporal pattern formation in chemical [1] and biological systems [2], and has recently seen renewed interest due to its possible significance in the dynamics of cortical processes [3–6]. In many cases, e.g., in the case of diffusive coupling between chemical oscillators, or in the case of resistive coupling between electrochemical oscillators, the coupling can be assumed to be linear in concentration differences or voltage differences. Generally, it is assumed that such diffusive interaction will eventually lead to synchronization of the phases of such oscillators. However, it has been argued before that diffusive interaction can also lead to dephasing of oscillators, which then entails a variety of new dynamical phenomena such as chemical turbulence [1] and intermittency [7].

The basic reason why dephasing can occur is that diffusive interaction between two limit cycle oscillators will in general shift these two oscillators away from their limit cycle, often such that one of the two oscillators will be pushed to the inside of the limit cycle, while the other one will be pushed to the outside of the limit cycle. If there is a strong radial gradient of the angular velocity, it can happen that the lagging oscillator will be pushed into a region of slower angular velocity and will thus be further delayed in its oscillation phase.

In general [1], at least for weakly coupled oscillators, the interaction between two oscillators at phases ϕ_1 and ϕ_2 can be expressed in terms of a phase interaction function $\Gamma(\delta\phi)$. As a result of interaction between the oscillators, the initial phase evolution equations $\dot{\phi}_1=1$ and $\dot{\phi}_2=1$ will be modified into

$$\dot{\phi}_1=1+\Gamma(\phi_1-\phi_2), \tag{1}$$

$$\dot{\phi}_2=1+\Gamma(\phi_2-\phi_1). \tag{2}$$

A negative derivative $\Gamma'(\delta\phi=0)<0$ means that the oscillators tend to synchronize, while a positive derivative $\Gamma'(\delta\phi=0)>0$ means that the oscillators tend to dephase. One fundamental problem in the study of coupled oscillator systems is the derivation of this phase interaction function from the dynamical equations describing the individual oscillators and from the interaction term. In general [1], the phase interaction function can be expressed as

$$\Gamma(\delta\phi)=\frac{1}{2\pi}\int_{\phi'=0}^{2\pi}\mathbf{Z}(\phi')\mathbf{p}(\phi',\delta\phi)d\phi', \tag{3}$$

where $p(\phi',\delta\phi)$ describes the perturbation of the state vector of an oscillator at phase ϕ' due to the interaction with another oscillator at phase $\phi'+\delta\phi$, and the sensitivity vector $Z(\phi')$, standing perpendicular to the isochrones, describes the phase change induced by a perturbation of an oscillator with phase ϕ'. Generally, the functions Z and p depend not only on the phases of the coupled oscillators, but also on the coupled oscillators' radial coordinates in phase space. For weakly coupled oscillators, however, radial position in phase space will in general not deviate very far from the limit cycle. One can then approximate the phase interaction function by evaluating the functions $Z(\phi')$ and $p(\phi',\delta\phi)$ at the location of the limit cycle that has the same phase ϕ'. By averaging the effect of the phase coupling over one limit cycle period in Eq. (3), this so-called phase approximation [1] then results in the phase interaction function Γ which is only a function of the phase difference $\delta\phi$.

In general, the integral in Eq. (3) has to be computed numerically. Analytical solutions are so far available only for the case of infinitesimally small limit cycles such as those that bifurcate out of a stationary point via a Hopf bifurcation [1]. In this paper, we want to discuss another limiting case in which the Γ function can be obtained analytically for finite-size limit cycles. The analytical solution will be useful to shed more light on the general conditions required to obtain dephasing phase interaction.

Using polar coordinates (r,θ), a nonlinear dynamical system with a limit cycle parametrized by $r_{\rm LC}(\theta)$ can be expanded in the vicinity of this limit cycle up to first order in $[r-r_{\rm LC}(\theta)]$ by

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$$\dot{r} = -[r - r_{\mathrm{LC}}(\theta)]\left(\frac{m_0}{2} + \sum_{l=1}^{\infty} m_l \sin(l\theta + \delta_l')\right), \quad (4)$$

$$\dot{\theta} = \omega(\theta) + [r - r_{\mathrm{LC}}(\theta)]\left(\frac{s_0}{2} + \sum_{l=1}^{\infty} s_l \sin(l\theta + \delta_l)\right). \quad (5)$$

In order to illustrate the role of variations of tangential phase velocity, we consider here the case where $r_{\mathrm{LC}}(\theta) = 1$ and uniform angular velocity $\omega(\theta) = 1$ on the limit cycle and uniform stability of the limit cycle $[m_l(\theta) = 2m\delta_{l,0}]$. We then obtain a limit cycle oscillator system given by

$$\dot{r} = -m(r-1), \quad (6)$$

$$\dot{\theta} = 1 + (r-1)\left(\frac{s_0}{2} + \sum_{l=1}^{\infty} s_l \sin(l\theta + \delta_l)\right). \quad (7)$$

Expressed in Cartesian coordinates (x,y), the linear interaction between the oscillators has a particularly simple form,

$$p_{\mathrm{Cartesian}}(x_1,y_1,x_2,y_2) = \begin{pmatrix} p_x(x_1,y_1,x_2,y_2) \\ p_y(x_1,y_1,x_2,y_2) \end{pmatrix} = \begin{pmatrix} -k_x(x_1 - x_2) \\ -k_y(y_1 - y_2) \end{pmatrix}, \quad (8)$$

which for $k_y = 0$ can be expressed in polar coordinates as

$$p(r_1,\theta_1,r_2,\theta_2) = \begin{pmatrix} p_r(r_1,\theta_1,r_2,\theta_2) \\ p_\theta(r_1,\theta_1,r_2,\theta_2) \end{pmatrix} = -k_x(r_1\cos\theta_1 - r_2\cos\theta_2)\begin{pmatrix} \cos\theta_1 \\ -\sin\theta_1 \end{pmatrix}. \quad (9)$$

The angular component of the sensitivity vector

$$Z(r,\theta) = \begin{pmatrix} Z_r(r,\theta) \\ Z_\theta(r,\theta) \end{pmatrix} \quad (10)$$

is simply

$$Z_\theta(r=1,\theta) = 1, \quad (11)$$

while the radial component will be calculated in the following. In general, the radial component indicates how much phase shift $\delta\phi$ will be caused by a radial perturbation of amplitude δr,

$$Z_r(r,\theta) = \frac{\delta\phi}{\delta r}. \quad (12)$$

If at time t_0 we apply a radial perturbation δr to a limit cycle oscillator at $r=1$ and $\theta = \theta_0$, this perturbation will decay exponentially as

$$r(t > t_0) = 1 + \delta r\; e^{-m(t-t_0)}. \quad (13)$$

The phase shift that the oscillator will have undergone can be calculated by integrating up how much the angular velocity will differ from 1 while the oscillator returns to the limit cycle

$$\delta\phi = \int_{t_0}^{\infty} (\dot{\theta} - 1)\,dt = \int_{t_0}^{\infty} \delta r e^{-m(t-t_0)} \times \left(\frac{s_0}{2} + \sum_{l=0}^{\infty} s_l \sin[l(t+\theta_0) + \delta_l]\right) dt. \quad (14)$$

Thus

$$Z_r(\phi) = \frac{\delta\phi}{\delta r} = \frac{s_0}{2m} + \sum_{l=1}^{\infty} \frac{s_l}{\sqrt{m^2 + l^2}} \sin\left(l\theta_0 + \delta_l + \arctan\frac{2}{m}\right). \quad (15)$$

In the limit of weak interactions, i.e., in the phase approximation, we can calculate the phase interaction function Γ by evaluating Z and p for $r=1$ with $\phi(r=1,\theta) = \theta$,

$$\Gamma(\delta\phi) = \Gamma_r(\delta\phi) + \Gamma_\theta(\delta\phi) = \frac{1}{2\pi}\oint Z_r p_r\, d\phi + \frac{1}{2\pi}\oint Z_\theta p_\theta\, d\phi, \quad (16)$$

obtaining

$$\Gamma_\theta(\delta\phi) = -\frac{1}{2} k_x \sin\delta\phi \quad (17)$$

and

$$\Gamma_r(\delta\phi) = \frac{k_x s_2/4}{\sqrt{m^2 + 2^2}}\left[-(1 - \cos\delta\phi)\sin\left(\delta_2 + \arctan\frac{2}{m}\right) + \sin\delta\phi\, \cos\left(\delta_2 + \arctan\frac{2}{m}\right)\right]. \quad (18)$$

An important consequence of this result is that only the second Fourier-expansion term of Eq. (7) has an influence on the phase interaction function. To make the formula easier to read, the asymmetric part of Γ_r can be written as

$$\Gamma_r^{\mathrm{asym}}(\delta\phi) = \frac{k_x}{2} \sin\delta\phi\; h(s_2,\delta_2,m), \quad (19)$$

where

$$h(s_2,\delta_2,m) = \frac{s_2}{2\sqrt{m^2+4}} \cos\left(\delta_2 + \arctan\frac{2}{m}\right). \quad (20)$$

In the special case that $\delta_2 + \arctan(2/m) = 0$, this function can be simplified to

$$h\left(s_2, \delta_2 = -\arctan\frac{2}{m}, m\right) = \frac{s_2}{2\sqrt{m^2+4}}. \quad (21)$$

In the case $\delta_2 = 0$, we obtain

$$h(s_2,\delta_2 = 0, m) = \frac{s_2}{2m + 8/m}, \quad (22)$$

and in the limit of large m, we obtain

$$h(s_2,\delta_2,m \gg 1) = \frac{s_2}{2m} \cos\delta_2. \quad (23)$$

In any case, the asymmetric part of the total phase interaction function can be written as

$$\Gamma^{\mathrm{asym}}(\delta\phi) = -\frac{1}{2}k_x[1-h(s_2,\delta_2,m)]\sin\delta\phi, \quad (24)$$

which, by analogous calculations for the case $k_y \neq 0$, takes the more general form

$$\Gamma^{\mathrm{asym}}(\delta\phi) = -\frac{1}{2}[(k_x+k_y)-(k_x-k_y)h(s_2,\delta_2,m)]\sin\delta\phi. \quad (25)$$

Thus the derivative $\Gamma'(\delta\phi=0)$ will have positive sign, meaning that the oscillators will dephase, if

$$1+\frac{k_y-k_x}{k_y+k_x}h(s_2,\delta_2,m)<0. \quad (26)$$

We now want to relate the above results with previous results obtained in the framework of the Ginzburg-Landau (GL) theory. According to the GL theory [1], in general, limit cycle oscillators near a Hopf bifurcation point can be described by a complex amplitude vector W which evolves in time according to

$$\frac{\partial W}{\partial t} = (1-ic_2|W|^2)W + (1+ic_1)(W-W') \quad (27)$$

if such an oscillator is coupled diffusively to another oscillator with amplitude vector W'. (The particular values of c_1 and c_2 reflect the characteristics of the original limit cycle system.) Further analysis [1] shows that diffusive interaction will lead to dephasing of these coupled oscillators if

$$1+c_1c_2<0, \quad (28)$$

which is known as the Benjamin-Feir (BF) criterion. It should be stressed that the GL equations on which the BF criterion is based represent a third order approximation of the dynamical system around its stationary point. From there it follows that the BF criterion takes into account only terms of up to this order.

Let us now consider the dynamical system

$$\dot{x} = -y\omega(x,y) - x\frac{r^2(x,y)-r_0^2}{2}m, \quad (29)$$

$$\dot{y} = x\omega(x,y) - y\frac{r^2(x,y)-r_0^2}{2}m, \quad (30)$$

with $r(x,y)=\sqrt{x^2+y^2}$ and

$$\omega(x,y) = 1+\frac{r^2(x,y)-r_0^2}{2}s_2 2xy. \quad (31)$$

This system contains nonlinear terms proportional to s_2 which are of fifth order in x,y. For $r_0=0$, this system undergoes a supercritical Hopf bifurcation, while for $r_0=1$, this system becomes equivalent up to $O((r-r_0)^2)$ to the system

$$\dot{x} = -y\omega(x,y) - x\frac{r(x,y)-1}{r(x,y)}m, \quad (32)$$

$$\dot{y} = x\omega(x,y) - y\frac{r(x,y)-1}{r(x,y)}m, \quad (33)$$

with

$$\omega(x,y) = 1+[r(x,y)-1]s_2\frac{2xy}{r^2(x,y)}, \quad (34)$$

which corresponds to the system of Eqs. (6) and (7) with only the second Fourier-expansion term retained.

If we transform the system of Eqs. (29) and (30) into the GL form, we obtain

$$c_1=0. \quad (35)$$

In other words, the BF criterion, which holds exactly for infinitesimal limit cycles, cannot be used reliably to derive the condition for desynchronization for finite-size limit cycles if the dynamical system contains nonlinear terms of higher order than 3. The fifth-order expansion terms that were considered in Eqs. (29) and (30), however, do not merely add numerical correction terms, but rather allow for a different desynchronization mechanism with different symmetry properties, as will be outlined in the following:

For phase oscillators which are coupled by linear attractive interaction, tangential perturbations $p_\phi(\theta)$ caused by this interaction always contribute to synchronizing the oscillators. Radial perturbations $p_r(\theta)$ are proportional to $\delta r \propto \sin(2\theta)$; if there is a uniform phase velocity gradient $Z_r(\theta)=\mathrm{const}$, periods where the oscillator is pushed into a region of faster phase velocity and periods in which the oscillator is pushed into a region of slower phase velocity will exactly cancel out. In order for radial perturbations to lead to desynchronization, either the limit cycle has to be skewed so that $\langle\delta r\rangle \neq 0$, or the phase gradient $Z_r(\theta)$ has to have a Fourier component of the same periodicity $\sin(2\theta)$ as the radial perturbation. These two cases are illustrated in Fig. 1. Figure 1(A) shows a skewed limit cycle with uniform phase gradient, for which the BF criterion [Eq. (28)] gives the conditions for desynchronization. Figure 1(B) shows a circular limit cycle with varying phase velocity gradient, for which the condition for desynchronization was derived in the present paper [Eq. (26)]. In a general case, both the skewedness of the limit cycle and the $\sin(2\theta)$ variation of the phase velocity gradient may add up in order for radial perturbations to dominate over synchronizing tangential perturbations.

Another important point to note is that terms that are of fifth order when one expands the dynamical system in a Taylor-expansion around the stationary point [Eqs. (29) and (30)] are actually the leading-order Fourier-expansion terms leading to desynchronization when one Fourier expands the dynamical system around a limit cycle of finite radius [Eqs. (6) and (7)].

As a final illustration, let us consider the well-studied Brusselator system [8], defined by

$$\dot{x} = F(x,y) = A-(B+1)x+x^2y, \quad (36)$$

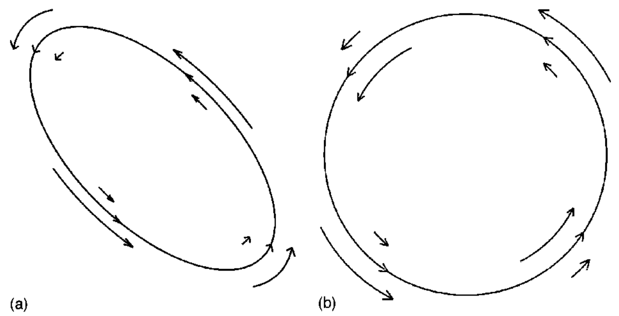

FIG. 1. (a) Schematic drawing of an elliptical limit cycle surrounded by a phase flow field whose radial gradient does not depend on the angle variable θ. The BF criterion will correctly predict that diffusive interaction in the x direction leads to dephasing if the gradient is sufficiently strong. (b) Schematic drawing of a circular limit cycle surrounded by a phase flow field whose radial gradient is proportional to $\sin 2\theta$. The effect of such variations in the angular velocity gradient are not taken into account by the BF criterion.

$$\dot{y} = G(x,y) = Bx - x^2 y. \tag{37}$$

This system has a stationary point at $(x_0, y_0) = (A, B/A)$, and will undergo a Hopf bifurcation at $B_c = 1 + A^2$. Transforming this dynamical system into Ginzburg-Landau form and performing and evaluating the Benjamin-Feir criterion yields [1] that diffusive coupling via the x variable should lead to dephasing if

$$\alpha = 1 - \frac{4 - 7A^2 + 4A^4}{6 + 3A^2} < 0 \tag{38}$$

or $A > 1.63 \ldots$. Since the Brusselator equations actually only contain terms up to third order in x, y, the BF criterion correctly predicts the conditions for dephasing even for values of parameter B away from the immediate vicinity of B_c.

Now consider the modified Brusselator system

$$\dot{x} = F(x,y) - (y - y_0)\,\omega(x,y), \tag{39}$$

$$\dot{y} = G(x,y) + (x - x_0)\,\omega(x,y), \tag{40}$$

with

$$\omega(x,y) = [(x - x_0)^2 + (y - y_0)^2]\, s\, (x - x_0)(y - y_0). \tag{41}$$

For this modified system, numerical calculations for $B = B_c + 0.01$, i.e., for limit cycles in the vicinity of the Hopf

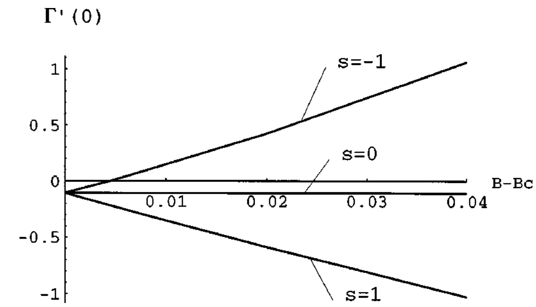

FIG. 2. Gradient of the phase interaction function $\Gamma'(0)$ as a function of $B - B_c$ for $A = 1.5$ and $s = -1, 0, 1$.

bifurcation, show that the gradient of the phase interaction function $\Gamma'(0)$ changes sign at $A = 1.63$ for $s = 0$, as predicted by the BF criterion. For $s = 1, -1$, however, the sign $\Gamma'(0)$ changes at $A = 2.29$ and $A = 1.38$, respectively. Figure 2 shows $\Gamma'(0)$ as a function of $B - B_c$ for $A = 1.5$ and $s = -1, 0, 1$. We see that the BF criterion which predicts $\Gamma'(0) < 0$ holds for the immediate vicinity of the Hopf bifurcation ($B = B_c$), but that $\Gamma'(0)$ can quickly change sign for $B > B_c$.

Model systems such as the Brusselator are valuable since they allow easy analytical manipulation while still allowing one to illustrate a wide variety of basic phenomena of nonlinear oscillator systems. The BF criterion owes much of its popularity to the fact that for such simple systems as the Brusselator system which only contain nonlinearities of up to third order, it holds exactly even for finite-size limit cycles. Limit cycle systems, such as the Hodgkin-Huxley neuronal oscillator that were derived to closely reproduce experimental observations, often contain nonlinearities of arbitrary order. This paper has shown that the dynamics of coupled oscillators with finite-size limit cycles cannot be reliably predicted by the BF criterion obtained from the third-order GL theory for infinitesimal oscillators. We have written down an expansion scheme for finite-size limit cycle oscillators, and used the analytical solution of a low-order truncation to find a scenario in which linear coupling can lead to dephasing. We hope that the present investigations will encourage other studies to analytically investigate the properties of finite size limit cycle oscillators rather than restricting analytical studies to infinitesimal limit cycle oscillators only.

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